Discrete Mathematics 112 (1993) 259–260
North-Holland

Note

Random edge domination

Peter Dolan* and Mark Halsey

Department of Mathematics, Bard College, Annandale-on-Hudson, N.Y. 12504, USA

Received 5 June 1990
Revised 5 December 1990

Abstract

Dolan. P. and M. Halsey, Random edge domination, Discrete Mathematics 112 (1993) 259–260.

The edge dominating number of the random graph $G_{n,1/2}$ is shown to be sharply concentrated.

In a graph, an *edge dominating set* is a subset of the edges with the property that every edge not in the subset is incident with at least one of the vertices in the dominating set. The *edge dominating number* ED of a graph is the order of the smallest set of edges in a dominating set. (Applications of edge domination are discussed in [5]). For ordinary graphs, the problem of finding a minimum edge dominating set is NP-complete. Indeed, the problem remains NP-complete for the class of bipartite graphs [2, 5].

We investigate the asymptotic behavior of this parameter in $G_{n,1/2}$, the random graph on n vertices with edge probability $\frac{1}{2}$. *Almost always* (with probability tending to 1 as n tends to infinity) its value will be one of two values. A similar phenomenon has been observed for the clique number [3, p. 51], and the vertex dominating number [4].

Theorem. *There exists a function $f(n)$ such that, as n tends to infinity,*

$$\Pr[\mathrm{ED}(G_{n,1/2}) = f(n) \text{ or } f(n)-1] \to 1.$$

Proof. There exists a function $\alpha(n) \sim 2 \ln n$ such that the largest independent set in $G_{n,1/2}$ almost always contains α or $\alpha+1$ vertices [3, p. 51]. Suppose $E = \{e_1, \ldots, e_k\}$

Correspondence to: Mark Halsey, Department of Mathematics, Bard College, Annandale-on-Hudson, N.Y. 12504, USA.

* Supported in part by an Asher B. Edelman Fellowship from Bard College.

are the edges of an edge dominating set. If $\{v_1, \ldots, v_t\}$ are the vertices of G that are not terminal vertices of an edge in E, then they must form an independent set since their internal edges would not be dominated. This means that almost always $t \leqslant \alpha + 1$ so $k \geqslant (n - \alpha - 1)/2$.

Consider $G_{n,1/2}$ as the union of two random graphs $G_1 = G_{n,p_1}$ and $G_2 = G_{n,p_2}$, where $p_2 = (\ln(n-\alpha)+\omega)/(n-\alpha)$, $\omega \to \infty$ arbitrarily slowly, and $p_1 = \frac{1}{2}[(n-2p_2)/(n-p_2)]$. Identify multiple edges, so the edge probability of $G_1 \cup G_2$ is $p_1 + p_2 - p_1 p_2 = \frac{1}{2}$. Since $p_1 < \frac{1}{2}$, G_1 almost always contains an independent set S of α vertices [3, p. 51]. There are two cases to consider:

Case 1: $n - \alpha$ *is even.*

The edges of G_2 almost always contain a perfect matching of the vertices of $G_1 \backslash S$ [1]. Furthermore, they almost never join any pairs of vertices of S since the expected number of such vertices is $\ll 1$. The edges of the matching will therefore almost always be an edge dominating set. Thus, in accordance with the lower bound, $\mathrm{ED}(G) = (n - \alpha)/2$.

Case 2: $n - \alpha$ *is odd.*

Form S' by removing a vertex from S. The edges of G_2 will almost always contain a perfect matching of the vertices of $G_1 \backslash S'$. The edges of the matching form a dominating set of cardinality $(n - \alpha + 1)/2$. Thus,

$$\frac{n-\alpha-1}{2} \leqslant \mathrm{ED}(G) \leqslant \frac{n-\alpha+1}{2}. \qquad \square$$

It should be noted that the independent edge dominating number (the size of the smallest maximal matching) is equal to the edge dominating number [5].

This theorem was originally part of a longer article. The authors thank the referees for their comments.

References

[1] P. Erdős and A. Rényi, On Random Matrices, Publ. Math. Inst. Hungar. Acad. Sci. 8 (1964) 455–461.
[2] Garey and D. Johnson, Computers and interactability, Bell Telephone Laboratories Inc., 1979.
[3] J. Spencer, Ten Lectures on the Probabilistic Method (SIAM, Philadelphia, PA, 1987).
[4] K. Weber, Domination number for almost every graph, Rostock Math. Kolloq. 16 (1981) 31–43.
[5] M. Yannakakis and F. Gavril, Edge Dominating Sets in Graphs, SIAM J. Appl. Math. 38 (1980) 364–372.